On the complexity group of stable curves

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Abstract

In this paper, we study combinatorial properties of stable curves. To the dual graph of any nodal curve, it is naturally associated a group, which is the group of components of the Néron model of the generalized Jacobian of the curve. We study the order of this group, called the complexity. In particular, we provide a partial characterization of the stable curves having maximal complexity, and we provide an upper bound, depending only on the genus g of the curve, on the maximal complexity of stable curves; this bound is asymptotically sharp for $g \gg 0$. Eventually, we state some conjectures on the behavior of stable curves with maximal complexity, and prove partial results in this direction.

Introduction

Combinatorics is often of great importance in the study of the moduli space of stable curves of genus g, $\overline{M}_g$. Recent examples are the combinatorial computation of its Euler characteristic [Las01]; the combinatorial aspects in the study of the Nef cone of $\overline{M}_g$ (see in particular question 0.13 of [GKM02]). The importance of combinatorics is evident also in the study of spin curves [CC03] [CCC07].

The main result relating the geometry of $\overline{M}_g$ to the combinatoric of stable curves is is the stratification of $\overline{M}_g$ by topological type, which is governed by the *weighted dual graph* associated to any stable curve (Definition 1.1). To any (multi)graph, it is naturally associated a group, which we will call *complexity group* of the graph (Definition 1.2). In particular, given a stable (or, more generally, nodal) curve C we call Δ_C the complexity group of the dual graph of the curve. This group has been extensively studied as an invariant of graphs, with applications to Physics, Chemistry, and many other areas, and it goes under many different names, such as critical group [Big99] [CR02], determinant group [BdlHN97], Picard group [BdlHN97], Jacobian group [BN07], abelian sandpiles group [CE02]. From the point of view of geometry, the complexity group was introduced in [Cap94], with the name of *degree class group*, in order to describe and handle the fibres of the compactification of the universal Picard variety $\overline{P}_{d,g}$ over $\overline{M}_g$ (also constructed in the same article). Moreover, this group arises naturally in the study of the compactified Jacobians of families of curves. More precisely, let R be a discrete valuation ring with fraction field K and residue field $\boldsymbol{k}$ and denote by B the spectrum of R. Consider a flat and proper regular curve $X \longrightarrow B$, such that the closed fiber $X_{\boldsymbol{k}}$ is geometrically irreducible. Under some technical assumptions, it is defined the group Φ of connected components of the the Néron model of the Jacobian J_K of the generic fiber X_K (see [BLR90] sec. 9.6). Notice that in this definition the closed fiber doesn't need to be nodal; when $X_{\boldsymbol{k}}$ is nodal, Φ is precisely $\Delta_{X_{\boldsymbol{k}}}$. Caporaso in [Cap08] gave a geometric counterpart of this construction, showing that, under the assumption that $(d-g+1, 2g-2) = 1$, there exists a space over $\overline{M}_g$ such that, for every regular family of stable curves over B, the Néron model of the Picard variety of degree d of X_K is obtained by base change via the moduli map $B \longrightarrow \overline{M}_g$. A yet another geometric interpretation of Φ is given by Chiodo in [Chi], where the r-torsion points of Φ are described as Néron models of r-torsion line bundles on X_K.

The relationship between the structure of this group and the structure of the graph has been studied, among others, by Lorenzini in several papers (see for instance [Lor90a] and [Lor90b]). Other important references are [Big99], [BdlHN97] and [CR02]. We addressed this problem in [BMS06], where in particular we constructed a family of graphs with cyclic complexity group. It seems that the question of finding

a relation between the structure of the complexity group and the geometry of the curve is extremely difficult and intricate.

Rather that in the structure, we are interested here in the cardinality of the complexity group. Kirkoff's Matrix Tree Theorem says that this integer is the number of spanning trees of the dual graph of the curve, usually called the *complexity*; this is the reason for our notation, also suggested by L. Caporaso in [Cap]. Moreover, it can be calculated as the determinant of a certain matrix (Theorem 1.4, Remarks 1.5 and 1.6).

We shall call complexity of a curve the corresponding complexity of its dual graph. The complexity being an upper semicontinuous function over $\overline{M}_g$ (Lemma 1.12), it defines a weak stratification on $\overline{M}_g$. In Section 1 we investigate the relationship between this stratification and the (strong) one given by topological type. This relation is also enlighted using the list of possible graphs for curves of genus 3 given in Section 2.

In particular, we are interested in the classes of curves with *maximal complexity*. Define the function $\psi(g) := \max\{|\Delta_C|,\ C \text{ stable curve of genus } g\}$; our guiding problems are the following.

1. Give a characterization of the curves C such that $c(C) = \psi(g)$, or at least with "big" complexity.

2. Find bounds for ψ, depending only on g.

In Section 3, we give partial answers to problem 1, finding necessary conditions for curves to have maximal complexity. We show that these curves are all graph curves in the sense of [BE91] i.e., they have simple trivalent dual graph Γ with $b_1(\Gamma) = g$; moreover, they have no disconnecting nodes (Theorem 3.2 and its combinatorial version Theorem 3.9). Using this reduction, we can apply some results of the immense literature regarding the complexity of regular graphs.

In Section 4 we give an answer to our second problem, using results of Biggs [Big74], McKay [McK83] and Chung-Yau [CY99]. In particular, we obtain an asymptotically sharp upper bound for $g \gg 0$. By what observed in the beginning, these bounds limit the number of connected components of the Néron model of the generalized Jacobian of stable curves. Note that this result is different from the one given by Lorenzini in [Lor93]: see Remark 4.3. In Section 4.1, we provide an example of families of graphs corresponding to stable curves with increasing genus, and compute explicitly the complexity depending on the genus, thus giving an explicit lower bound.

In the last section we discuss the conjectural behavior of the curves with maximal complexity. In particular, it seems that the graph of such a curve should have maximal connectivity (which is 3 in this case). Moreover, it seems that the girth has to be big (Conjecture 5.1 and 5.6). We prove in this section some partial results that seem to support the conjectures (Proposition 5.2 and Corollary 5.3). Moreover, we prove a uniform necessary condition holding for any sequence of graphs of curves with maximal complexity (Theorem 5.7), using again a result of McKay.

Even though our point of view is irreparably geometrical, our main results are proven with (very simple) combinatorics methods, and can be rewritten in purely combinatorial terms (see in particular 3.1). We would like to express the hope that our geometric approach does not discourage non-geometers from the reading of the article; and in particular from considering the list of open question given in section 5.

Acknowledgments

We would like to express our gratitude to L. Caporaso for introducing us the problems we are dealing with in the present paper and for following this work very closely and to C. Casagrande for the important corrections and suggestions she gave us.

We also thank A. Machi for transmitting us much encouragement after reading a preliminary version.

1 Combinatorial invariants of nodal curves

We work over an algebraically closed field $\boldsymbol{k} = \overline{\boldsymbol{k}}$. A *nodal* curve is a reduced curve which has only ordinary double points as singularities.

The dual graph of a curve

Definition 1.1. *To a nodal curve C we can associate a graph Γ_C, called the weighted dual graph, given by a triple (V, E, g), where V is the set of vertices, E the set of edges, and g a function on the set V with non-negative integer values. This triple is defined in the following way*

- *to each irreducible component A corresponds a vertex*[1] v_A*;*
- *to each node intersecting the components A and B (where A and B can coincide) corresponds an edge connecting the vertices v_A and v_B;*
- $g : V \longrightarrow \mathbb{Z}_{\geq 0}$ *is the function that associates to any vertex v the geometric genus of the corresponding component of C.*

Call γ the number of irreducible components of C and δ the number of nodes. Thus Γ_C has γ vertices, δ edges, and among the edges there is a loop for every node lying on a single irreducible component of C. The weighted graph encodes all the topological information about the curve. Of course, conversely, any weighted graph can be realized as the dual graph of a nodal curve.

It is important to stress that dual graphs of nodal curves can have more than one edge connecting two nodes, and can also have loops. These are usually called *multigraphs*. In this paper, by graph we will always mean multigraph, while a graph without loops and multiple edges will be called *simple*.

Given a graph Γ with δ edges, γ vertices and c connected components, its first Betti number is $b_1(\Gamma) := \delta - \gamma + c$; it corresponds to the number of independent cycles on Γ.

Let $\{C_1, \ldots, C_\gamma\}$ be the set of irreducible components of a nodal curve C with δ nodes and c connected components. Recall that the arithmetic genus of C can be computed by the following formula

$$g(C) = \sum_{i=1}^{\gamma} g(C_i) + \delta - \gamma + c = \sum_{v \in V(\Gamma_C)} g(v) + b_1(\Gamma_C). \tag{1.1}$$

By analogy, for any weighted graph Γ given by the triple (V, E, g), we will call $g(\Gamma) := \sum_{v \in V} g(v) + b_1(\Gamma)$ the *(arithmetic) genus of the graph*.

Complexity group

Let us consider a connected nodal curve C. Let $\{C_i\}_{i=1,\ldots,\gamma}$ be the irreducible components of C. Define

$$k_{ij} := \begin{cases} \sharp(C_i \cap C_j) & \text{if } i \neq j \\ -\sharp(C_i \cap \overline{C \setminus C_i}) & \text{if } i = j \end{cases}$$

As $C_i \cap \overline{C \setminus C_i} = \bigcup_{j \neq i} C_i \cap C_j$, we have that, for fixed i, $\sum_j k_{ij} = 0$. For every i set

$$\underline{c}_i := (k_{1i}, \ldots, k_{\gamma i}) \in \mathbb{Z}^\gamma.$$

Call $Z := \{\underline{z} \in \mathbb{Z}^\gamma : |\underline{z}| = 0\}$. As observed before, $\underline{c}_i \in Z$. Let us call Λ_C the sublattice of Z spanned by $\{\underline{c}_1, \ldots, \underline{c}_\gamma\}$. In fact, Λ_C is a lattice in Z (it has rank $\gamma - 1$) as we will show in a moment (see [Cap08] for a geometric proof of this fact).

[1] Sometimes in the literature the vertices of a graph are denoted "nodes"; of course we will never adopt this notation which is extremely confusing in our context. A node will always be for us an ordinary double point of a curve!

Definition 1.2. *The complexity group of C is the finite abelian group $\Delta_C := Z/\Lambda_C$.*

It is important to notice that this group depends *only* on the dual (non-weighted) graph of the curve; clearly it is defined for any connected[2] graph. As noted in the Introduction, this group arises in many contexts where graphs are used, and it is known with many other names.

Let M be the $\gamma\times\gamma$ matrix whose columns are the $\underline{c}_i$'s. We will call M the *intersection matrix*, the name clearly deriving from its geometrical meaning. However, in literature, M is known as the *(combinatorial) Laplacian* matrix (cf. e.g. [Bol98] and [Lor91]). It is obtained from the so-called adjacency matrix of Γ_C subtracting the vertex degrees on the diagonal.

Complexity of a graph

A tree is a connected graph G with $b_1(G) = 0$. Let Γ be a graph. A *spanning tree* of Γ is a subgraph of Γ which is a tree having the same vertices as Γ.

Definition 1.3. *The complexity of Γ, indicated by the symbol $c(\Gamma)$, is the number of spanning trees contained in Γ (see e.g. [Big74], sec 6, [Ber70], cap.3 $\natural$ 5, [Wes96], sec 2.2).*

Observe that $c(\Gamma) = 0$ if and only if Γ is not connected, and that if Γ is a connected tree $c(\Gamma) = 1$. For the complexity of the dual graph associated to a curve C, we will often use the symbol $c(C)$, instead of $c(\Gamma_C)$. The following theorem, known as Kirkoff's Matrix Tree Theorem, will be a key ingredient for our work. There are at least three different proofs of this result; see [BMS06] for a proof and for the references.

Theorem 1.4. (Matrix Tree Theorem) *Let $s, t \in \{1, \dots \gamma\}$. Using the above notations, if $M^\star$ is obtained from M by deleting the t-th column and the s-th row, then*

$$c(\Gamma) = (-1)^{s+t+\gamma-1} det(M^\star).$$

In particular, the Matrix Tree Theorem assures that, in the case of the dual graph of a *connected* curve C, the matrix M has rank $\gamma - 1$ i.e., Λ_C is indeed a lattice.

Remark 1.5. For $r \in \{1, \dots, \gamma\}$, consider the isomorphism $\alpha_r : Z \xrightarrow{\sim} \mathbb{Z}^{\gamma-1}$ which consists of deleting the r-th component. The group Δ_C is the quotient of $\mathbb{Z}^{\gamma-1}$ by the lattice generated by

$$\underline{c}'_i := (k_{1i}, \dots, \widehat{k_{ri}}, \dots, k_{\gamma i}).$$

Observe that again $\sum_i \underline{c}'_i = \underline{0} \in \mathbb{Z}^{\gamma-1}$. Therefore Δ_C is presented by the matrix $M^\star$ obtained from M by deleting a column and the r-th row. Hence, we can compute its cardinality via Theorem 1.4

$$c(\Gamma_C) = |\Delta_C| = |\det(M^\star)|.$$

So, we can conclude that *the cardinality of the complexity group of a curve C is the complexity of the dual graph Γ_C.*

Remark 1.6. By the diagonalization theorem of integer matrices (i.e. the structure theorem for finite abelian groups, see [Art91]), M is equivalent over $\mathbb{Z}$ to a diagonal matrix $diag(d_1, d_2, \dots, d_\gamma)$, where the $d'_i s$ are the invariant factors, i.e. the non-negative integers obtained in the following way: Let D_i be the greatest common divisor of the $i \times i$ minors of M. Then $d_i = D_i/D_{i-1}$ (cf. also [Lor89], Theorem 1.5). Note that $d_\gamma = 0$ and $d_i > 0$ for $i \neq \gamma$, so $\Delta_C = \oplus_{i=0}^{\gamma-1}\mathbb{Z}/d_i\mathbb{Z}$. In particular, $|\Delta_C|$ is equal to the greatest common divisor of all the $(\gamma - 1) \times (\gamma - 1)$ minors of M.

On the other hand, if we diagonalize M over the real numbers, we get real eigenvalues $0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_\gamma$. These eigenvalues are deeply studied in Combinatorics. Note that in general the λ_i's have no relation with the invariant factors, not even if they happen to be integers; a nice counterexample can be found in Section 9.2 of [BdlHN97]. An attempt to construct the invariant factors from the $\lambda'_i s$, for a particular class of graphs, can be find in [CR02].

Let us note that in particular $|\Delta_C| = \gamma^{-1}\lambda_2\lambda_3 \dots \lambda_\gamma$ (cf. [Big74] cor.6.5).

[2] The definition could be easily extended to non connected graphs/curves.

Stable curves and their graphs

Definition 1.7. *A curve C of genus $g \geq 2$ over $\boldsymbol{k}$ is stable (resp. semistable) if it is nodal, connected and such that if $D \subset C$ is a smooth rational component, then $|D \cap \overline{C \setminus D}| \geq 3$ (resp. ≥ 2).*

The moduli space of stable curves of genus g, $\overline{M}_g$, is a projective variety of dimension $3g-3$, the so-called Deligne-Mumford compactification of the moduli space of smooth curves of genus g, M_g. The theory of stable curves was first introduced by A. Mayer and D. Mumford and was first developed in [DM69] in order to prove the irreducibility of M_g in any characteristic. In that paper, the authors prove the main properties of stable curves used later by D. Gieseker in [Gie82] to establish the existence of $\overline{M}_g$.

Given a graph Γ and a vertex v, the degree $d(v)$ of v is the number of half edges touching v. The combinatorial version of the definition of a stable curve is the following.

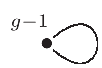
Definition 1.8. *A weighted graph Γ of arithmetic genus $g \geq 2$ is stable if*

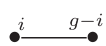
$$2g(v) - 2 + d(v) > 0 \quad \textit{for any} \quad v \in V. \tag{1.2}$$

Γ is said to be semistable if the inequality above holds with $\geq$.

Topological stratification of $\overline{M}_g$

There is a natural stratification on $\overline{M}_g$ given by topological type. Each stratum of codimension k is the subset of $\overline{M}_g$ consisting of classes of stable curves with weighted graphs of genus g having exactly k edges. In particular, the stratum of codimension 0 is the open set $M_g \subset \overline{M}_g$ of smooth curves. The strata of codimension 1 are $[g/2]+1$, and the corresponding graphs are of the following types.





for $i = 1, \ldots, [g/2]$.

The closure of these strata are effective divisors on $\overline{M}_g$, the so-called boundary divisors, usually denoted by Δ_i, for $i = 0, \ldots, [g/2]$, according to the preceding list. For instance, Δ_0 has as generic point corresponding to an irreducible curve of geometric genus $g-1$ with exactly one node. It can be described as the locus of curves having at least one non-disconnecting node. The boundary of $\overline{M}_g$ is $\partial\overline{M}_g = \overline{M}_g \setminus M_g = \cup_{i=0}^{[g/2]} \Delta_i$.

The union of the strata of codimension k is the variety of isomorphism classes of stable curves having exactly k nodes; its closure is the locus of curves with at least k nodes.

Degeneration of nodal curves

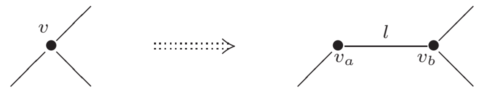
The following well-known result describes the possible transformations on a dual graph corresponding to the degenerations of a nodal curve.

Proposition 1.9. *Let $\Gamma = (V, E, g)$ be a weighted graph. Let C be a nodal curve having Γ as dual weighted graph. The dual graphs of the possible nodal curves obtained by degenerations of C are obtained from Γ via a sequence of the following transformations:*

(I) given a vertex v such that $g(v) \geq 1$ add a loop on v and decrease its weight by 1;

(II) given a vertex v, and given two nonnegative integers a and b such that $a + b = g(v)$, substitute it with two vertices v_a and v_b with weights respectively a and b and one edge l connecting them.

The figure below illustrates operation (II).



Of course, there are many ways to perform operation (II) (see also Definition 3.3 below). Notice that this is the opposite operation of contracting an edge, in the sense that if we contract the new edge l we get the original graph.

Note also that if our given curve C is stable, degenerations of C obtained by performing operation (I) are still stable, while if we perform operation (II) this is the case only if $2a - 2 + d(v_a) \geq 1$ and $2b - 2 + d(v_b) \geq 1$.

Polygonal curves

We present here an elementary combinatorial proof of the following well-known fact for stable curves. A geometric proof can be obtained using the above results about the topological stratification of $\overline{M}_g$.

Lemma 1.10. *Let C be a stable curve of genus $g \geq 2$. Then*

1. *C has at most $3g - 3$ nodes and $2g - 2$ irreducible components.*
2. *Assume that C has $3g-3$ nodes. Then C has $2g-2$ components $C_1, \ldots, C_{2g-2}$ and, if $\nu_i : C_i^\nu \longrightarrow C_i$ is the normalization of C_i, then $C_i^\nu \simeq \mathbb{P}^1$ and $|\nu_i^{-1}(C_i \cap C_{sing})| = 3$ for all i.*

Proof. Let $\Gamma = (V, E, g)$ be the weighted graph associated to C. Note that $\sum_{v \in V} d(v) = 2\delta$, hence we can rewrite formula (1.1) as

$$g = \sum_{v \in V} \left(g(v) + \frac{d(v)}{2} \right) - \gamma + 1. \tag{1.3}$$

By the connectedness of C, $d(v) \geq 1$ for any v. Using also the stability condition, we have that $g \geq \sum_{v \in V} \frac{3}{2} - \gamma + 1 = \frac{\gamma}{2} + 1$, hence, $\gamma \leq 2g - 2$. Now, using (1.1) again, we have that $\delta \leq 3g - 3$, and (1) follows.

Now, if $\delta = 3g - 3$, again by (1.1), we see that necessarily $\gamma = 2g - 2$ and $g(v) = 0$ for any $v \in V$. Moreover, using again (1.3), we see that $d(v) = 3$ for any vertex v, so also (2) is proved. $\square$

Stable curves with $3g - 3$ nodes (and $2g - 2$ components), are the 0-strata of the topological stratification of $\overline{M}_g$; they are rigid, in the sense that any deformation of such curves in a family of stable curves must have necessarily at least one node smoothed. Indeed, from Proposition 1.9 we see that both operations (I) and (II) cannot be performed, i.e. there is no possible further degeneration for such curves. We will call such curves *polygonal curves* (also called large limit curves, see [Tyu03]). Polygonal curves with simple graph are called graph curves ([BE91]).

We refer to [BE91] for an ample discussion on the importance and the properties of these curves. Let us just make a couple of remarks. The associated dual graphs are trivalent (multi)graphs such that the weight function g is 0 on every vertex. Due to the fact that $\mathbb{P}^1$ with 3 fixed points has no non-trivial automorphisms, the automorphism group of such curves coincides with the automorphism group of the graph. Artamkin in [Art05] has given a recursive rule that computes the number

$$\sum_{\substack{\Gamma \ trivalent, \\ b_1(\Gamma) = g}} \frac{1}{|Aut(\Gamma)|},$$

which is the "stacky top self-intersection" of the boundary divisor $\partial \overline{M}_g$.

Stratification by complexity of $\overline{M}_g$

We can define a function $c\colon \overline{M}_g \longrightarrow \mathbb{Z}_{\geq 0}$ associating to every $[C] \in \overline{M}_g$ its complexity $c(C)$.

Of course, c is bounded from above, i.e., there is a bound for the complexity of a stable curve of given genus; indeed, by Lemma 1.10, there is only a finite number of possible graphs. In section 4 we provide upper bounds, which are asymptotically sharp for $g \gg 0$.

Remark 1.11. Clearly, this wouldn't make sense for nodal curves (not even for semistable ones). Indeed, blowing up a node an arbitrary number of times doesn't change the genus of the curve, but it can increase arbitrarily the complexity of a nodal curve (see for instance [BMS06] Proposition 3.3.).

Lemma 1.12. *The function $c\colon \overline{M}_g \longrightarrow \mathbb{Z}_{\geq 0}$ is upper semi-continuous.*

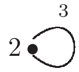
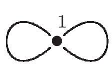
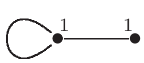
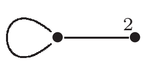
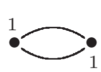
Proof. Follows from Proposition 3.4 below. □

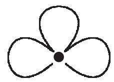
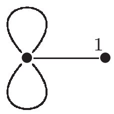
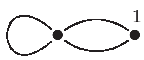
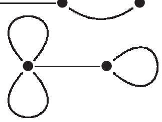
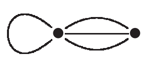
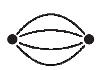
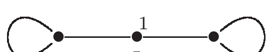
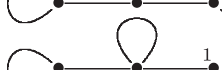
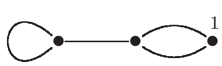
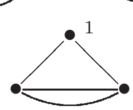
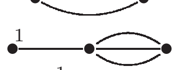
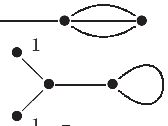
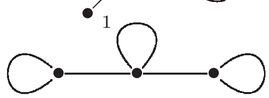
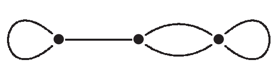
This result implies in particular that we could define a stratification "by complexity" of $\overline{M}_g$. This is clearly a much rougher stratification than the one by topological type; *the set of curves in $\overline{M}_g$ with given complexity is a (maybe empty) union of components of different codimension strata of the topological type stratification, of different codimension* (see the case of genus 3 in the next section).

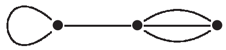
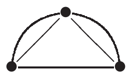
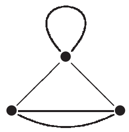
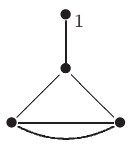
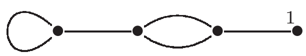
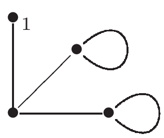
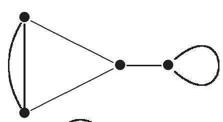
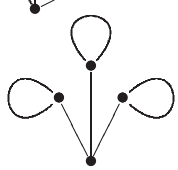
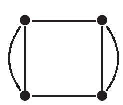
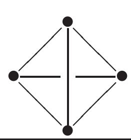
For instance, the set $\overline{M}_g^{c=1}$ of curves with complexity one is the set of curves whose dual graph is a tree with loops. It contains the curves of compact type $\overline{M}_g^{ct} = \overline{M}_g \setminus \Delta_0$, but also the interior of Δ_0, and other strata of bigger codimension; it also contains 0-strata, i.e. isolated points (see next section). The complement $\overline{M}_g \setminus \overline{M}_g^{c=1}$ of curves with complexity greater than one is a closed subset of codimension 2.

2 List of graphs for $\overline{M}_3$

In this section, we list all the possible weighted graphs for stable curves of genus 3, as well as their complexity and their complexity group. This list (with one error, now corrected, which was kindly pointed out to us by A. Chiodo) appeared in [BMS06]. We will use $\mathbb{Z}_n$ to denote the quotient group $\mathbb{Z}/n\mathbb{Z}$. The graphs are ordered by increasing the number of nodes (i.e. according to the codimension of corresponding stratum of the stratification by topological type). In the graphs we will indicate the weight of each vertex only if it is not zero. Recall that if C is a stable curve of genus 3, then C has at most 6 nodes and 4 components.

Graph configuration	Nodes	Components	Complexity	DCG
	0	1	1	0
	1	1	1	0
	1	2	1	0
	2	1	1	0
	2	2	1	0
	2	2	1	0
	2	2	2	$\mathbb{Z}_2$
	2	3	1	0

Graph configuration	Nodes	Components	Complexity	DCG
	3	1	1	0
	3	2	1	0
	3	2	1	0
	3	2	2	$\mathbb{Z}_2$
	3	2	3	$\mathbb{Z}_3$
	3	3	1	0
	3	3	1	0
	3	3	2	$\mathbb{Z}_2$
	4	2	1	0
	4	2	2	$\mathbb{Z}_2$
	4	2	3	$\mathbb{Z}_3$
	4	2	4	$\mathbb{Z}_4$
	4	3	1	0
	4	3	1	0
	4	3	2	$\mathbb{Z}_2$
	4	3	5	$\mathbb{Z}_5$
	4	3	3	$\mathbb{Z}_3$
	4	4	1	0
	5	3	1	0
	5	3	2	$\mathbb{Z}_2$

Graph configuration	Nodes	Components	Complexity	DCG
	5	3	3	$\mathbb{Z}_3$
	5	3	8	$\mathbb{Z}_8$
	5	3	5	$\mathbb{Z}_5$
	5	4	5	$\mathbb{Z}_5$
	5	4	2	$\mathbb{Z}_2$
	5	4	1	0
	6	4	2	$\mathbb{Z}_2$
	6	4	5	$\mathbb{Z}_5$
	6	4	1	0
	6	4	12	$\mathbb{Z}_{12}$
	6	4	16	$\mathbb{Z}_4 \times \mathbb{Z}_4$

Note that the set of curves of genus 3 with complexity 1 contains at least one stratum of the topological stratification of any codimension. The set of curves with complexity 2 contains at least one stratum among the ones of codimension greater or equal to 2. The set of curves with complexity 6 is empty.

3 Stable curves with maximal complexity

We will now focus our interest in curves with maximal complexity. We can see that in case $g = 3$ above, the curves reaching the maximal complexity are polygonal curves. However, as already observed, there are also polygonal curves with small, even 0, complexity. In what follows we prove that, indeed, any curve with maximal complexity is necessarily a polygonal curve and, moreover, it is a graph curve without disconnecting nodes. In Section 5 we shall make some remarks and conjectures on sufficient conditions.

We shall need the following well-known result, whose proof is elementary.

Proposition 3.1. *Let Γ be a graph. If e is an edge of Γ which is not a loop, call $\Gamma - e$ the graph obtained from Γ by removing e, and $\Gamma \cdot e$ the one obtained by contracting e. Then, we have the following relation between the complexities of these three graphs:*

$$c(\Gamma) = c(\Gamma - e) + c(\Gamma \cdot e). \tag{3.4}$$

See [BMS06] for a geometric interpretation and discussion of this proposition.

The main result of this section is the following

Theorem 3.2. *Let C be a stable curve of genus $g \geq 3$ with maximal complexity. Then, C is a curve without disconnecting nodes and with trivalent simple dual graph. In particular, C is a graph curve.*

From a geometrical point of view, this result implies in particular that the curves with maximal complexity lie on those 0-strata of the topological stratification which are contained in $\partial\overline{M}_g \setminus \cup_{i=1}^{[g/2]} \Delta_i$.

See the end of this section for a purely combinatorial statement.

Our strategy is the following: we consider a generic stable curve C of genus g and its weighted dual graph, Γ_C; then, if Γ_C is not as stated in Theorem 3.2, we modify the graph obtaining a new weighted graph Γ', which is the dual graph of another stable curve of genus g with the desired properties, and we prove that $c(\Gamma_C) < c(\Gamma')$. So, we need to perform operations on the graph increasing strictly the number of spanning trees. A. Kelmans made a deep study of operations increasing the complexity of a graph, even though from a different point of view; see for instance [Kel76].

Recall the two operations associated to the degeneration of a curve described in Proposition 1.9. We now study when do these operations increase the complexity of the graph.

Definition 3.3. *We shall distinguish two different kinds of operation (II):*

(II)a if the new edge l is a disconnecting edge;

(II)b otherwise, i.e. if there is a cycle $\mathcal{C}$ that contains v such that $\mathcal{C} \cup l$ is a cycle for the new graph.

Proposition 3.4. *Let R be a discrete valuation ring with fraction field K and residue field $\boldsymbol{k}$, and $f\colon X \longrightarrow \operatorname{spec} R$ a family of nodal curves. Let Γ_K, resp. $\Gamma_{\boldsymbol{k}}$, be the weighted graph associated to the generic fiber X_K, resp. to the closed fiber $X_{\boldsymbol{k}}$.*

Then, the complexity function $c\colon \operatorname{spec} R \longrightarrow \mathbb{Z}_{\geq 0}$ is constant if and only if $\Gamma_{\boldsymbol{k}}$ is obtained from Γ_K via a sequence of operations of type (I) and (II)a. Otherwise, $c(\Gamma_{\boldsymbol{k}}) > c(\Gamma_K)$.

Proof. Clearly, the weights on the graphs do not interfere with the complexity, as well as adding loops.

We now prove that, applying operation (II) the complexity remains the same only in case (II)a, and it increases strictly in case (II)b. Call Γ' the graph obtained by applying operation (II) to a vertex v of degree d, and, accordingly to the notation of Proposition 1.9, call d_a and d_b the vertex degrees of v_a and v_b, respectively, in Γ'. Clearly

$$d = d_a + d_b - 2,$$

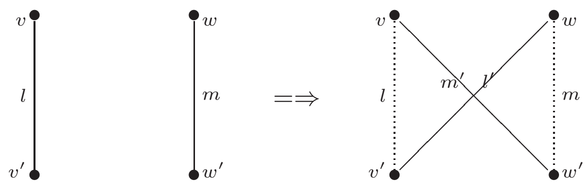
Since we are adding a vertex and a component, the first Betti number of the graph, b_1, remains the same. However, the complexity can increase, but not decrease. Indeed, as observed above, we have that $\Gamma = \Gamma' \cdot l$. So, by formula (3.4),

$$c(\Gamma') = c(\Gamma) + c(\Gamma' - l) \geq c(\Gamma).$$

Moreover, we see from the above formula that the complexity remains the same if and only if $c(\Gamma' - l) = 0$, i.e., if and only if l is a disconnecting edge for Γ'. $\square$

The switching of two edges

As we will often make use of another operation on graphs, it is convenient to describe it separately. Let us consider a graph Γ, and fix two distinct edges l and m, with associated vertices v, v' and w, w' respectively. Let us suppose that $v \neq w$ (but we do not ask that $v \neq v'$ or $w \neq w'$, i.e. l and m can be loops). We shall construct a new graph Γ' which has the same vertices as Γ, such that $\Gamma \setminus \{l, m\}$ is a subgraph of Γ', and the edges l, m, are substituted by l' and m' as illustrated in figure below.

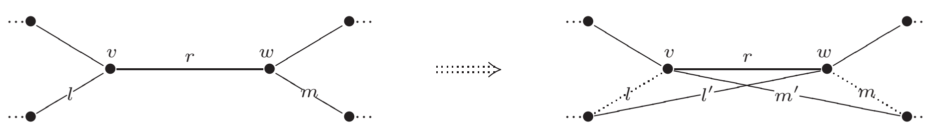


So, the new edges l' and m' connect v and w', v' and w respectively. We will call this process *the switching of* l *and* m *with respect to* v *and* w[3]. Note that the vertex degrees of Γ and Γ' are the same.

In general, this operation doesn't increase the complexity; counterexamples are easy to construct. However, we shall prove that, for certain graph configurations, it does.

Proposition 3.5. *Let C be as in Theorem 3.2. Then C has no disconnecting nodes.*

Proof. Suppose that C has a disconnecting node, which corresponds in $\Gamma = \Gamma_C$ to a disconnecting edge r. Consider l, m two edges adjacent to r. The only case in which we cannot find two different edges, is if one of the vertex joined by r, say v, has degree one. In this case, as the curve C is stable, necessarily $g(v) \geq 1$; we therefore modify the graph by adding a loop on v, and decreasing by one the weight on v. We still have a graph associated to a stable curve of genus g with the same complexity, which we will call again Γ, and we can choose l and m as above. Now, we perform the switching of l and m w.r.t. v and w:[4]



Note that $\Gamma \cdot r = \Gamma' \cdot r$. Now, Γ' is the dual graph of a stable curve (indeed, we can think of the modification made as if we have resolved two nodes of the curve, and attached the components where they belong in a different way).

Now, if the edge r is still a disconnecting edge for Γ', then both l and m should be also disconnecting edges in Γ. In this case, we must have $g(v) \geq 1$ and we perform operation (I) in v. Then, if we choose l to be the new loop, after the switching of l and m with respect to v, the edge r will not be diconnecting anymore.

So, we can suppose that r is no longer a disconnecting edge for Γ', and it is immediate to check that we have introduced no new disconnecting edges, i.e. if f is a disconnecting edge for Γ', then it is also a disconnecting edge for Γ.

[3]Note that this operation is the $\widetilde{X}$-transformation described in [Tsu96].
[4]this operation is the "slide transformation of l and m at v and w along r" with the terminology of [Tsu96].

Now, applying Proposition 3.1 and observing that $c(\Gamma - r) = 0$, while $c(\Gamma' - r) > 0$, we get:

$$c(\Gamma') = c(\Gamma' \cdot r) + c(\Gamma' - r) = c(\Gamma \cdot r) + c(\Gamma' - r) > c(\Gamma \cdot r) = c(\Gamma).$$

This proves the statement. $\square$

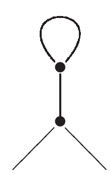
Remark 3.6. In [JR], sec.3, it is observed that a general graph (not necessarily regular) has to be free of disconnecting edges in order to have maximal complexity. Proposition 3.5, as stated, could be indeed proven in the same way. However, the operation used by the two authors is the switching of *one* edge, which modifies the vertex degrees; as we will see, it is necessary for us to remain in the class of cubic graphs, that's why we apply our switching.

Proposition 3.7. *Let C be as in Theorem 3.2. Then C is a loopless graph curve.*

Proof. From Proposition 3.5 we can suppose that C has no disconnecting nodes. Let Γ be the weighted graph associated to C and suppose Γ is not a loopless trivalent graph. Using the degeneration operations described in Definition 1.9, we will describe an algorithm in two steps that, given Γ with no disconnecting nodes, produces a loopless graph Γ_3 with strictly bigger complexity than Γ, such that $b_1(\Gamma_3) = g$ and each vertex has degree 3 and weight 0. Therefore Γ_3 is the dual graph associated to a loopless graph curve of genus g.

FIRST STEP (Reduction to a curve having only rational components): We replace any vertex v_i with strictly positive weight $g(v_i)$ with a bouquet given by a vertex of weight 0 and $g(v_i)$ loops attached to it. This is a reiterate application of operation (I) and does not change the arithmetic genus, nor the complexity. Call Γ_1 the graph obtained in this way.

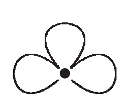
SECOND STEP (Reduction to a loopless graph curve): suppose that in Γ_1 there is a vertex v with degree d greater or equal to 4. As Γ_1 is obtained from Γ by adding loops, it has no disconnecting edges. We can therefore perform operation (II)b on v, with the request that $\deg(v_a) \geq 3$, and $\deg(v_b) \geq 3$ (i.e. remaining in the class of graphs of stable curves). As proved in Proposition 3.4, this operation increases strictly the complexity. Moreover, the degrees of v_a and v_b are strictly smaller than the degree of v. We perform this transformation until all the vertices of the new graph have exactly degree 3. Call Γ_2 the resulting graph. So by the genus formula Γ_2 has $2g-2$ vertices of weight 0, $3g-3$ edges, and $b_1(\Gamma_2) = g$; equivalently, Γ_2 is the graph associated to a graph curve. Moreover, Γ_2 has no disconnecting edges, and no loops. Indeed, if there were a loop, Γ_2 would contain a subgraph of the form:



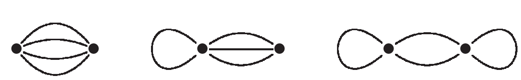
.

hence with a disconnecting node. Moreover, by what observed above, $c(\Gamma_2) > c(\Gamma_1)$. $\square$

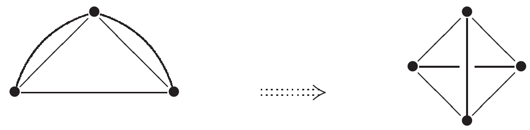
Example 3.8. Let us consider the dual graph of a smooth genus 3 curve; this is just one vertex with weight 3. Let us apply the above algorithm to this graph. With the first step, we obtain the following "bouquet" of 3 loops,



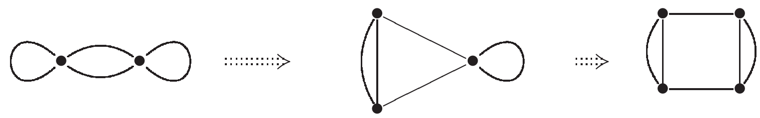
which is the graph of a stable curve with a single irreducible rational component with 3 nodes. As the irreducible component has geometric genus 0, we apply the second step. We have several choices. So, we can obtain either one of the following configurations.



Notice that all these graphs have still vertices with degree greater then 3. So, we therefore must go on applying step 2. For example, one of the possibilities for the first graph would be the following.

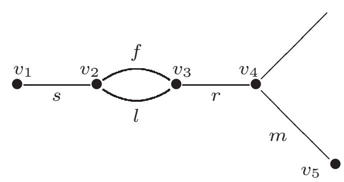


If we start from the third graph, we can perform the following chain of transformations (indeed, in this case, these operations are forced).

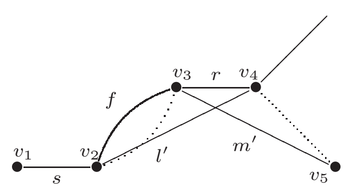


As also the simple example above shows, the algorithm of Proposition 3.7 doesn't give a unique output. On the contrary, if we start from a bouquet of g loops we can obtain *all* the possible cubic graphs without disconnecting edges using step 2, as one can see in the following way. Consider a cubic graph without disconnecting edges Σ with $b_1(\Sigma) = g$. Let T be a spanning tree for Σ. To contract all the edges of T is the reverse operation to step 2 of the algorithm, and the result is the bouquet of g loops.

Proof. (*of Theorem 3.2*) By Proposition 3.5 and 3.7 we can suppose that $\Gamma = \Gamma_C$ is trivalent, loopless and without disconnecting edges. Hence, the only possible multiple edges are of the form



Note that l and m have to be distinct as well as v_1 and v_4 (otherwise Γ would have disconnecting edges). With the notations adopted in the figure, we apply the switching of l and m w.r.t. v_3 and v_4:



We have constructed a new trivalent graph Γ' which has one less couple of multiple edges: indeed, it is immediate to check that we have not created any other multiple edges.

Let us prove that $c(\Gamma') > c(\Gamma)$. We prove it by giving an injective map from the spanning trees of Γ to the spanning trees of Γ', as follows.

Let T be a spanning tree of Γ. If $r \in T$, the switching of Γ we performed transforms T into a spanning tree T' of Γ', with $r \in \Gamma'$.

Now, suppose $r \notin T$. We shall distinguish between 4 different situations.

- $l, m \notin T$ ($\Rightarrow f \in T$). T is transformed into a tree T' of Γ' with $l' \notin T', m' \notin T', r \notin T'$ and $f \in T'$.

- $l \in T$, $m \notin T$ ($\Rightarrow f \notin T$). Then, in T, the path connecting v_2 and v_4 passes by l. So, the switching of l and m along r restricted to T is not a spanning tree of Γ' since l' would create a cycle containing l' and v_3 and v_5 get disconnected. So, we consider the subgraph of Γ' obtained by the union of m' with the switching of T minus l'. It is easy to see that this is a spanning tree T' of Γ', with $l' \notin T'$, $m' \in T$, $r \notin T'$ and $f \notin T'$.

- $l \notin T$, $m \in T$ ($\Rightarrow f \in T$). In this case we must do a further distinction.

  - In T the path connecting v_2 and v_4 does not contain m.
    In this case T is transformed into a spanning tree T' of Γ' with $l' \notin T'$, $m' \in T'$, $r \notin T'$ and $f \in T'$.

  - In T the path connecting v_2 and v_4 contains m.
    In this case the switching we performed restricted to T is not a spanning tree of Γ', since m' creates a cycle and v_4 and v_2 get disconnected. So, we consider T' as the union of l' with the switching of Γ restricted to T minus m'. Then T' is a spanning tree of Γ' with $l' \in T'$, $m' \notin T'$, $r \notin T'$ and $f \in T'$.

- $l, m \in T$ ($\Rightarrow f \notin T$). Also in this case we have to distinguish between the following situations:

  - In T the path connecting v_2 and v_4 does not contain m.
    Then the switching of l and m along r restricted to T is not a spanning tree of Γ' since it creates a cycle containing l' and v_4 and v_5 get disconnected. So if we consider T' as the union of f with the switching of T minus s, then T' is a spanning tree of Γ' with $l' \in T'$, $m' \in T'$, $r \notin T'$ and $f \in T'$.

  - In T the path connecting v_2 and v_4 contains m.
    In this case the switching we performed restricted to T is a spanning tree T' of Γ' with $l' \in T'$, $m' \in T'$, $r \notin T'$ and $f \notin T'$.

So, the association of each spanning tree T of Γ with T' gives an injective map from the spanning trees of Γ into the spanning trees of Γ'.

Now, to prove that, indeed strict inequality holds, we observe that Γ' has at least one spanning tree T' such that l', f and r are not in T' ($\Rightarrow m' \in T'$), and this kind of spanning trees are not in the image of the map constructed above. $\square$

3.1 Combinatorial version

We can give a purely combinatorial translation of the above results, as follows. Recall that a graph Γ is said to be k-connected if for any set S of $k-1$ edges of Γ, $\Gamma \setminus S$ is still connected. Γ is said to be *strictly* k-connected if it is k-connected but not $k+1$-connected. So, a graph is 1-connected if and only if it is connected; it is strictly 1-connected if and only if it is connected and it has at least one disconnecting node. Of course, a trivalent graph can never be 4-connected (any triple of edges incident in one vertex is a disconnecting set).

Given a positive integer g, let $\mathcal{G}_g$ be the set of all weighted multigraphs of genus g satisfying condition (1.2). Theorem 3.2 can now be written in the following way.

Theorem 3.9. *The graphs $\Gamma \in \mathcal{G}_g$ reaching the maximal complexity are trivalent, simple and 2-connected.*

We shall call a simple trivalent graph a *cubic graph*.

4 Bounds on the maximal complexity of stable curves

The question of an upper bound on the complexity depending on the genus has several geometrical meanings: for instance, it implies that there is a bound for the irreducible components of the compactified

Jacobian of a stable curve, and a bound for the group of components of the Néron model of the relative Jacobian of a family of curves having C as special fiber; see also Remark 4.3 . Let us define

$$\psi(g) := \max\{c(C), C \text{ stable curve of genus } g\}.$$

Remark 4.1. Notice that ψ is strictly increasing with respect to the genus g. In fact, if C is a curve of genus g such that $\psi(g) = |\Delta_C|$, let C' be the stable curve obtained from C by adding an extra node connecting two of the components of C. Then, using the above formula for the genus of C', one gets $g_{C'} = g + 1$ and, clearly, $c(C') > c(C)$.

According to the results of the preceding section, we know that curves achieving maximal complexity are in particular graph curves. Using this fact, we can find a first rough bound:

$$\psi(g) \leq \binom{3g-3}{g} \leq 2^{3g-4}.$$

For the proof, see [Lor90b], Lemma 2.7. Note that the first inequality follows immediately from the fact that any spanning tree in a cubic graph of order $2g-2$ is obtained by removing g edges.

Applying the results on the surjectivity of the "Abel-Jacobi map" in [BN07], we could derive a better bound:

$$\psi(g) \leq \binom{2g-2}{g}.$$

Both these bounds are not sharp, even for low genus. Using a result of Biggs [Big74], we can prove the following.

$$c(\Gamma) \leq \frac{1}{2g-2}\left(\frac{6g-6}{2g-3}\right)^{2g-3} =: \alpha(g). \tag{4.5}$$

For $g = 3$, this bound is optimal. However, the bound is not asymptotically sharp.

The complexity of k-regular graphs has been studied also by McKay [McK83], and Chung-Yau [CY99]. We can apply their results obtaining a sharper bound.

Theorem 4.2. *Let C be a stable curve of genus g, and let Γ be its dual graph. Then*

$$c(\Gamma) \leq \frac{2\ln(2g-2)}{3(2g-2)\ln(9/8)} \exp\left(\frac{12}{\sqrt{\pi}}\left(\frac{\ln 9/8}{\ln(2g-2)}\right)^{5/2}\right)\left(\frac{4}{\sqrt{3}}\right)^{2g-2} =: \beta(g). \tag{4.6}$$

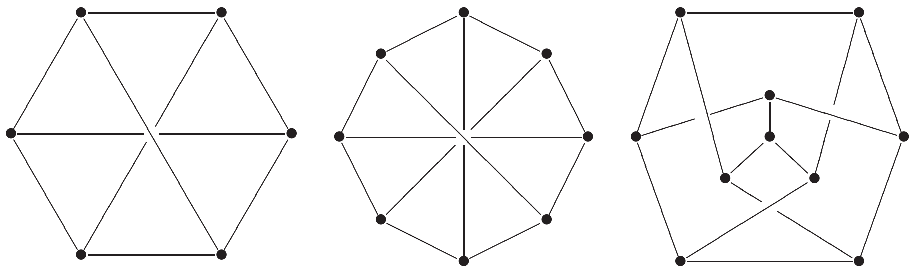
Moreover, this bound is asymptotically sharp for $g \gg 0$.

Proof. By making explicit the computations in Theorem 4 of [CY99], we obtain the bound (4.6) for 3-regular graphs with $2g-2$ vertices. The thesis is now straightforward because of the reduction to graph curves of Theorem 3.2. McKay proves in [McK83] that there is a sequence of cubic graphs $\{\Gamma_i\}_{i\in\mathbb{N}}$ with increasing orders n_i's, such that, if we set

$$\tau(\Gamma_i) := (n_i c(\Gamma_i))^{\frac{1}{n_i}}, \tag{4.7}$$

then $\tau(\Gamma_i) \longrightarrow \frac{4}{\sqrt{3}}$ for $i \to \infty$. Indeed, he proves much more than the existence, but that a "random" sequence of cubic graphs satisfies this property, see also [Lyo05]. Hence, the constant $\frac{4}{\sqrt{3}}$ in the bound (4.6) is the best possible. ∎

In the next picture we describe the graphs with maximal complexity for $g = 4$, $g = 5$ and $g = 6$, respectively.



$\Gamma_1, g = 4$ $\Gamma_2, g = 5$ $\Gamma_3, g = 6$

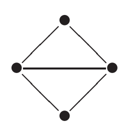
Note that Γ_3 is the famous Peterssen graph. These graphs are proved to be of maximal complexity among simple cubic graphs of their respective orders in [JR], sec.5.

Remark 4.3. As already noticed, the geometric meaning of this result is that it gives a bound on the group of connected components of the Néron model of the degree-d Picard variety for families of *stable* curves ([BLR90], Theorem 1, sec.9.6), and as well on the number of irreducible components of the fibres of the scheme P^d_g constructed in [Cap08] and of $\overline{P_{d,g}}$ of [Cap94]. Of course this is also a bound on the cardinality of the group of components of the Néron models of the Jacobians of such families. However, notice that this bound is different from the ones found by Lorenzini in [Lor93] (see also [BLR90], Theorem 9 sec.6.9). Indeed, given any strictly henselian discrete valuation ring R, with algebraically closed residue field $\boldsymbol{k}$ and field of fractions K, any regular family of curves $f: X \longrightarrow \operatorname{spec} R$ such that the *general* fiber X_K is of genus g, and the Jacobian J_K has *potential good reduction*, Lorenzini finds a bound, depending only on g, for the group of components of the Néron model of f. We obtain instead a bound for the group of components of the Néron model of *any* smooth family $f: X \to \operatorname{spec} R$ such that the *closed* fiber $X_{\boldsymbol{k}}$ is a stable curve. So one could say that his approach is dynamic, in the sense that it depends on the generic fiber, while our approach is static, as it starts from a given curve. Moreover, the boundedness comes on the one hand from the assumption of potential good reduction on the general fiber, on the other from the assumption of stability of the special one. As it is proven in [Lor90a], a family has potential good reduction if and only if the closed fiber has a *tree* as dual graph, hence complexity 1.

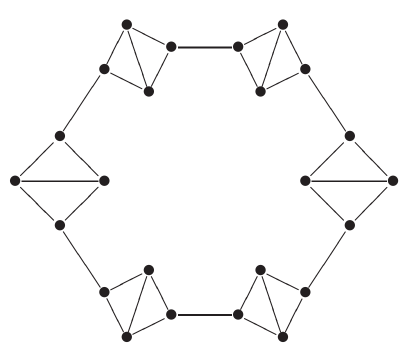
4.1 Example and lower bound

We present here an example of a family of cubic graphs of increasing order and increasing complexity. In particular, this gives explicit lower bounds on $\psi(g)$ (even though not sharp, given the above result).

Let Γ_m be the graph with $4m$ vertices of valency 3 formed from m pairwise disjoint graphs G_i of the following form:



by adding m edges $l_1, \ldots, l_m$ to link them as a ring, as shown in the figure below for $m = 6$. Clearly, Γ_m is the dual graph of a graph curve of genus $g = 2m + 1$.

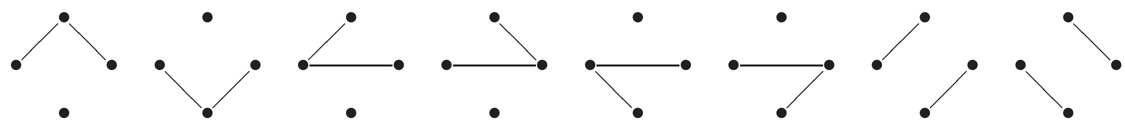


Proposition 4.4. $c(\Gamma_m) = 2m8^m$.

Proof: To prove this formula we will proceed by hands counting the number of spanning trees for Γ_m. Notice that the complexity of the subgraphs G_i's is 8.

Choosing one of the edges l_i, the number of spanning trees not containing it is 8^m. Indeed, all the others l_j's have to be included in any spanning tree, while for any G_k, we have to count the 8 possibilities for the spanning trees. So, we have $m8^m$ of these.

On the other hand, if T is a spanning tree that contains all the l_i, then there is one and only one j such that $T \cap G_j$ is disconnected, and it has to be of one of the 8 following forms:



For $i \neq j$, $T \cap G_i$ is a spanning tree for G_i as above. Therefore, we have $8m8^{m-1} = m8^m$ possible spanning trees of this form.

So, summing all up, we have $2m8^m$ spanning trees, as required. □

Notice that

$$c(\Gamma_m) = (g-1)8^{\frac{g-1}{2}} = (g-1)(2\sqrt{2})^{g-1}.$$

So, for odd g, we have this lower bound on ψ. For even g, remembering that ψ is an increasing function of the genus, we have $\psi(g) > \psi(g-1)$. Hence, ψ is bounded from below by $(g-2)(2\sqrt{2})^{g-2}$.

5 Further results and conjectures

3-connectedeness

Note that a trivalent graph with loops, or with double edges is strictly 2-connected. With Theorem 3, we have therefore excluded the strictly 1-connected case, and some of the cases of strict 2-connectedness. ¿From this, and from the observation of graphs with maximal complexity for low genus, it is natural to ask

Conjecture 5.1. *A cubic graph with maximal complexity is* 3*-connected.*

This seems to be generally believed, also for bigger classes of graphs (see e.g.[JR] sec.3), but no proof is known. Let us note that 3-connected graph curves have also interesting geometrical properties; indeed, they are precisely those graph curves for which the canonical bundle is very ample, yielding an embedding morphism [BE91].

With similar techniques to the ones used in Section 3, we can prove a partial result.

Proposition 5.2. *Let C be as in Theorem 3.2. Then*

1. *Γ_C has no couple of disconnecting edges that lay on a cycle of length ≤ 6.*
2. *Γ_C has no couple of disconnecting edges such that at least one of them is adjacent to a cycle of length ≤ 4.*

Corollary 5.3. *Conjecture 5.1 holds for $g \leq 8$.*

¿From a result on the so-called "Abel-Jacobi map" for graphs proved on [BN07] (Theorem 1.8), we can derive the following

Proposition 5.4 (Baker-Norine)**.** *If Γ is any k-connected graph of order n,*

$$c(\Gamma) \geq \binom{n}{k-1}.$$

This proves that the complexity grows with the connectivity. In the case of cubic graphs, however, the bound obtained for 3-connected graphs is just quadratic in the order.

Maximal girth

The explicit sequences of cubic graphs with large complexity have all *large girth*. Let us explain the terminology. The *girth* of a graph is the length of the shortest cycle. A simple argument shows that in a cubic graph the girth cannot exceed $2\ln n/\ln 2$ (n being the order of the graph). Hence, the girth of a family of cubic graphs can grow at most as the rate of the logarithm of the number of vertices.

Definition 5.5. *A sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ of cubic graphs with increasing orders n_i is called* sequence of large girth *if*

$$\lim_{i\to\infty}\frac{\ln(n_i)}{\ln 2\; girth(\Gamma_i)} \quad \textit{is finite.}$$

McKay proves in [McK83] that sequences with large girth satisfy condition (4.7).

Conjecture 5.6. *A sequence $\{\Gamma_i\}_{i\in\mathbb{N}}$ of cubic graphs with maximal complexity has large girth.*

See [JR], sec.3 for an heuristic argument supporting this conjecture.

From a result of McKay we can derive the following property of sequences of trivalent graphs of maximal complexity. Given a graph Γ, and a positive integer m, let $C_\Gamma(m)$ be the number of cycles of length $\leq m$ in Γ.

Theorem 5.7. *Let m be a positive integer. For any $\epsilon > 0$ there exist a positive integer j such that: given any infinite sequence of trivalent graphs $\{\Gamma_i\}_{i\in\mathbb{N}}$ such that Γ_i has $2i$ vertices, and reaches the maximal complexity between cubic graphs of order $2i$, the following inequality holds*

$$\frac{C_i(m)}{2i} \leq \epsilon \quad \textit{for any } i \geq j,$$

where $C_i(m) = C_{\Gamma_i}(m)$.

Proof. Theorem 4.5 of [McK83] states that, given a sequence of cubic graphs $\{\Gamma_i\}_{i\in\mathbb{N}}$ satisfying (4.7), then, for each fixed m,

$$\frac{C_i(m)}{n_i} \to 0 \quad \text{for } i \to \infty.$$

This does not directly imply the statement, which is uniform, in the sense that the constant j depends only on m and not on the chosen sequence. However, let us argue as follows: given m, consider a sequence of cubic graphs with maximal complexity *and* with maximal cardinality of $C(m)$. Clearly this sequence satisfies condition (4.7). Now, by assumption, the statement holds uniformly on every sequence of graphs of maximal complexity. □

Note that a sequence of graphs of large girth satisfies trivially the condition of Theorem 5.7.

Further speculations

Other open questions are:

- Are cubic graphs of maximal complexity *strongly regular*?
- Do cubic graphs of maximal complexity have *maximal automorphism group*?
- Is a cubic graph with maximal complexity and given genus *unique*?
- Let C be a graph curve with maximal complexity; suppose that it is indeed 3-connected. Then the canonical map is an embedding realizing C as a configurations of lines in $\mathbb{P}^{g-1}$. We conjecture that these lines are in "general position", in a sense that can be made precise. This property seems to be connected with the large girth property.

- Let Γ be a cubic graph with maximal complexity; suppose that it is indeed 3-connected. We think that the "Clifford index" of this graph, as defined for instance by Bayer and Eisenbud, has to be maximal as well.
- Let Γ be a cubic graph with maximal complexity; suppose that it is indeed 3-connected. We wonder if any cutset of 3 edges is made of adjacent edges.

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